MA2160 – Final Exam
Spring 2006

Non-Calculator Section

1. Evaluate $\int_0^1 \frac{1}{x^2+1}\,dx$.

$$\int_0^1 \frac{1}{x^2+1}\,dx = \arctan x \Big|_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4}$$

$\frac{\pi}{4}$ (5 pts)

2. Evaluate $\int_1^2 2x(\ln(x))\,dx$.

$$2\int_1^2 x\ln x\,dx = 2\left[\frac{x^2\ln x}{2}\Big|_1^2 - \int_1^2 \frac{x}{2}\,dx\right] = x^2\ln x\Big|_1^2 - \frac{x^2}{2}\Big|_1^2$$

$u = \ln x$ $\quad v = \frac{x^2}{2}$

$du = \frac{1}{x}$ $\quad dv = x\,dx$

$$= 4\ln 2 - 2 + \frac{1}{2}$$

$$= 4\ln 2 - \frac{3}{2}$$

$4\ln 2 - \frac{3}{2}$ (5 pts)

3. Evaluate $\int_4^\infty \frac{1}{(x-1)^2}\,dx$.

$$\lim_{a\to\infty}\int_4^a (x-1)^{-2}\,dx = \lim_{a\to\infty} -(x-1)^{-1}\Big|_4^a$$

$$= \lim_{a\to\infty}\left[\frac{-1}{a-1} - \frac{-1}{4-1}\right]$$

$$= \lim_{a\to\infty}\frac{-1}{a-1} + \frac{1}{3}$$

$$= \frac{1}{3}$$

$\frac{1}{3}$ (5 pts)

4. Calculate the integral $\int \frac{1}{(x+2)(x+3)}\,dx$.

$$\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$1 = A(x+3) + B(x+2)$$

$$1 = Ax + 3A + Bx + 2B$$

$$0x + 1 = (A+B)x + 3A + 2B$$

$A + B = 0$

$A = -B$

$3A + 2B = 1$

$-3B + 2B = 1$

$-B = 1$

$B = -1$

$A = 1$

$$\int \frac{1}{x+2}\,dx + \int \frac{-1}{x+3}\,dx = \ln|x+2| - \ln|x+3| + C$$

$\ln\left|\frac{x+2}{x+3}\right| + C$ (5 pts)

5. Find the solution $y(x)$ to the differential equation $x\frac{dy}{dx} = 4y$ subject to initial condition $y(1) = 3$.

$$\int \frac{dy}{y} = \frac{4\,dx}{x}$$

$$\ln|y| = 4\ln|x| + C_1$$

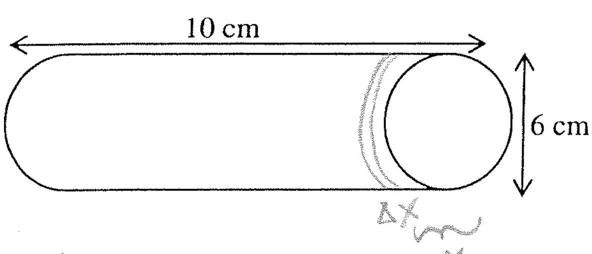
$$|y| = e^{4\ln|x| + C_1} = C_2 e^{4\ln|x|} = C_2 x^4$$

$$y = C_3 x^4$$

$$3 = C_3$$

$y = 3x^4$ (5 pts)

6. Find the volume of the solid cylinder below using integration.



$$\int_0^{10} 9\pi\,dx = 9\pi x\Big|_0^{10} = 90\pi$$

90π (5 pts)

7. Find the sum of the series $-1+\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\ldots$

$a = 1 \quad -\left(1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}\cdots\right) = \frac{-a}{1-x} = \frac{-1}{1+\frac{1}{2}}$

$x = \left(\frac{-1}{2}\right)$

$= -\frac{2}{3}$

$-\frac{2}{3}$ (5 pts)

8. Write the 2[nd] degree Taylor polynomial approximation to the function $x^{\frac{3}{2}}$ for values of x near $x=4$. **Show your work.**

$f(x) = x^{3/2}$ $\qquad f(4) = 8$

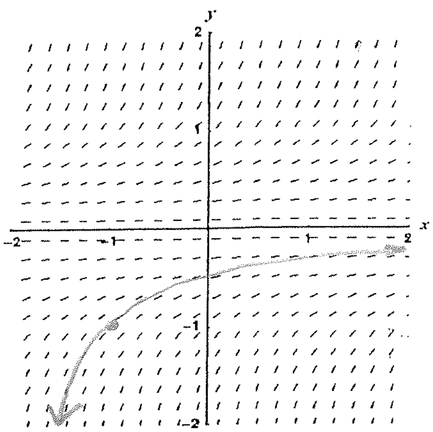
$f'(x) = \frac{3}{2}x^{\frac{1}{2}}$ $\qquad f'(4) = 3$

$f''(x) = \frac{3}{4}x^{-1/2}$ $\qquad f''(4) = \frac{3}{8}$

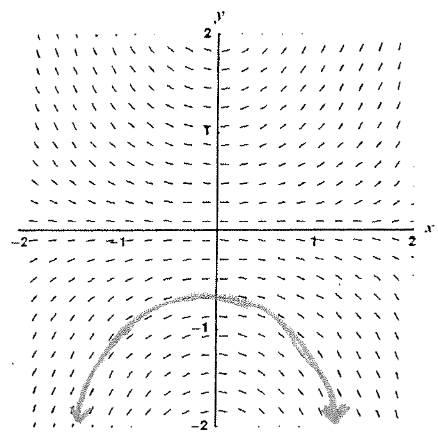
$8 + 3(x-4) + \frac{3}{8}\frac{(x-4)^2}{2!}$ (5 pts)

9. Consider the slope fields given below.

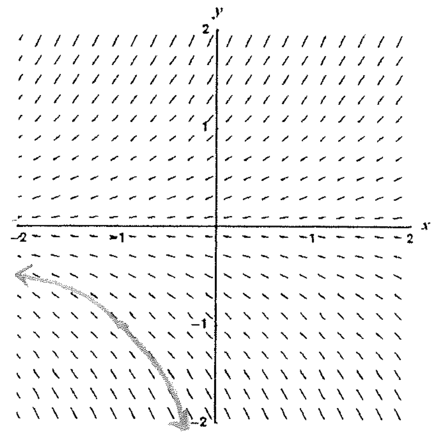
a) Sketch the solution curve through the point $(-1,-1)$ for each of the following slope fields. Extend each curve as far to the left and right as possible. (6 pts)



Slope field A



Slope field B



Slope field C

b) Match the slope fields from part (a) with the following differential equations: (6 pts)

Differential equation	Slope field
$y' = y$	C
$y' = y^2$	A
$y' = xy$	B

10. The function $f(x) = e^x$ is approximated by a Taylor polynomial of degree 3 about $x = 0$ on the interval $[0,1]$. Use the Lagrange error bound to find an upper bound for the approximation error.

$f^{(4)}(x) = e^x \Rightarrow M = e < 3$

$|E_3| \leq \frac{3|1|^4}{4!} = \frac{3}{8}$

$\frac{3}{8}$ (5 pts)

11. Trying to model the response to a stimulus, psychologists use the Weber-Fechner Law. This law states that the rate of change of a response, r, with respect to a stimulus, s, is inversely proportional to the stimulus. Model this law as a differential equation. Solve this differential equation with the initial condition that $r(s_0) = r_0$. Simplify your answer.

$\frac{dr}{ds} = \frac{k}{s}$

$\int \frac{1}{k} dr = \int \frac{ds}{s}$

$\frac{r}{k} = \ln|s| + C_1$

$r = k\ln|s| + C_2$

$r(s_0) = r_0$

$\Rightarrow r_0 = k\ln|s_0| + C_2$

$C_2 = r_0 - k\ln|s_0|$

$\Rightarrow r = k\ln|s| + r_0 - k\ln|s_0|$

$r = r_0 + k\ln\left|\frac{s}{s_0}\right|$

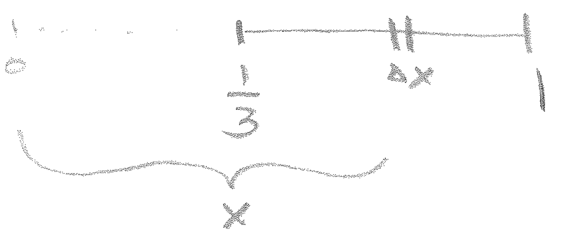
$r = r_0 + k\ln\left|\frac{s}{s_0}\right|$ (5 pts)

12. Use Euler's method with $y(0)=10$ as the initial condition with 2 steps to estimate y when $x=1$ for the differential equation $\frac{dy}{dx}=y+x$.

x	y	m	Δy
0	10	10	5
$\frac{1}{2}$	15	15.5	7.75
1	22.75		

$y(1) \approx 22.75$ (5 pts)

13. Two like magnetic poles repel each other with a force $F=\frac{k}{x^2}$ newtons, where k is a constant. Express the work needed to move them along a line from $\frac{1}{3}$ meters apart to 1 meter apart.



$$W = \int F\,d$$

$$= \int_{\frac{1}{3}}^{1} \frac{k}{x^2}\,dx = k\int_{1/3}^{1} x^{-2}\,dx$$

$$= -\frac{k}{x}\Big|_{\frac{1}{3}}^{1} = -k + 3k = 2k$$

$2k$ joules (5 pts)

14. Consider the vectors $\vec{u} = 3\vec{i} + 2\vec{j} + \vec{k}$ and $\vec{v} = \vec{i} - \vec{j} + \vec{k}$.

a) Find $\vec{u} \times \vec{v}$.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix} = (2-(-1))\hat{i} - (3-1)\hat{j} + (-3-2)\hat{k}$$

$$= 3\hat{i} - 2\hat{j} - 5\hat{k}$$

$3\hat{i} - 2\hat{j} - 5\hat{k}$ (4 pts)

b) Find a unit vector in the direction of $\vec{u}$.

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{3^2+2^2+1^2}}$$

$\frac{3}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{\hat{k}}{\sqrt{14}}$ (4 pts)

c) Compute the cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{(3-2+1)}{\sqrt{14}\sqrt{1^2+(-1)^2+(1)^2}} = \frac{2}{\sqrt{14 \cdot 3}}$$

$\frac{2}{\sqrt{42}}$ (4 pts)

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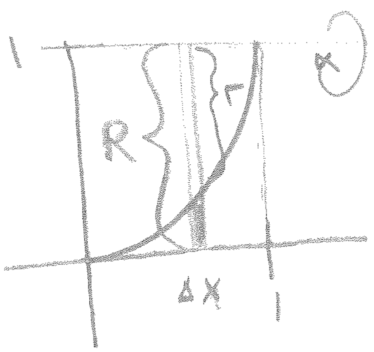
Calculator Section

1. Write in the form $ax+by+cz=d$ an equation of a plane which is perpendicular to the vector $3\vec{i}+2\vec{j}-\vec{k}$ and contains the point $(0,1,-1)$.

$$3(x-0)+2(y-1)-1(z-(-1))=0$$
$$3x+2y-2-z-1=0$$
$$3x+2y-z=3$$

$3x+2y-z=3$ (5 pts)

2. Consider the region bounded by the curves $y=x^2, x=1$, and the x-axis from $x=0$ to $x=1$. Find the volume of the solid obtained by rotating this region abut the line $y=1$.



$$\pi\int_0^1 (1)^2-(1-x^2)^2\,dx$$
$$\pi\int_0^1 1-(1-2x^2+x^4)\,dx$$
$$\pi\int_0^1 2x^2-x^4\,dx = \frac{7\pi}{15}$$

$\frac{7\pi}{15}$ (5 pts)

3. A can of soda pop is put into a refrigerator that is kept at $35^{\circ}F$. The cooling of the can is modeled by Newton's law of cooling $\frac{dh}{dt} = -k(h-35)$. Solve the differential equation if the can was at $80^{\circ}F$ when it was placed in the refrigerator and its temperature after 10 minutes is $50^{\circ}F$.

$$\int \frac{dh}{h-35} = \int -k\,dt$$

$$\ln|h-35| = -kt + C_1$$

$$|h-35| = C_2 e^{-kt}$$

$$h-35 = C_3 e^{-kt}$$

$$h = 35 + C_3 e^{-kt}$$

$$h(0) = 80 \implies 80 = 35 + C_3$$

$$C_3 = 45$$

$$h(t) = 35 + 45e^{-kt}$$

$$h(10) = 50 \implies 50 = 35 + 45e^{-10k}$$

$$\frac{15}{45} = e^{-10k}$$

$$-10k = \ln\left(\frac{1}{3}\right)$$

$$k = \frac{\ln\left(\frac{1}{3}\right)}{-10} \approx .1099$$

$h(t) = 35 + 45e^{-.1099t}$ (6 pts)